Expanding the denominator as an infinite series, we get

$$
\begin{aligned}
V(x,s) &= f(s)\left[e^{-x\sqrt{src}}+e^{-(2h-x)\sqrt{src}}\right]\left[1-e^{-2h\sqrt{src}}+e^{-4h\sqrt{src}}-e^{-6h\sqrt{src}}+\ldots\right] \\
&= f(s)\left\{\left[e^{-x\sqrt{src}}-e^{-(2h+x)\sqrt{src}}+e^{-(4h+x)\sqrt{src}}+\ldots\right]+\left[e^{-(2h-x)\sqrt{src}}-e^{-(4h-x)\sqrt{src}}+\ldots\right]\right\} \\
&= f(s)\left\{e^{-x\sqrt{src}}+\sum_{n=1}^{\infty}(-1)^n e^{-(2hn+x)\sqrt{src}}+\sum_{n=1}^{\infty}(-1)^{n-1}e^{-(2hn-x)\sqrt{src}}\right\} \qquad (51)
\end{aligned}
$$

Substituting $f(s)=\frac{V_0}{s}$ for a step input and taking the inverse transform [12],[9]

$$
\begin{aligned}
v(x,t) &= V_0\left\{\left[erfc\left(\sqrt{\frac{R_xC_x}{4t}}\right)-erfc\left(\sqrt{\frac{R_{2h+x}C_{2h+x}}{4t}}\right)+erfc\left(\sqrt{\frac{R_{4h+x}C_{4h+x}}{4t}}\right)+\ldots\right]\right. \\
&\quad+\left.\left[erfc\left(\sqrt{\frac{R_{2h-x}C_{2h-x}}{4t}}\right)-erfc\left(\sqrt{\frac{R_{4h-x}C_{4h-x}}{4t}}\right)+\ldots\right]\right\} \\
&= V_0\left\{erfc\left(\sqrt{\frac{R_xC_x}{4t}}\right)+\sum_{n=1}^{\infty}(-1)^n erfc\left(\sqrt{\frac{R_{2nh+x}C_{2nh+x}}{4t}}\right)\right. \\
&\quad+\left.\sum_{n=1}^{\infty}(-1)^{n-1}erfc\left(\sqrt{\frac{R_{2nh-x}C_{2nh-x}}{4t}}\right)\right\} \qquad (52)
\end{aligned}
$$

where $R_{2nh\pm x}=(2nh\pm x)r$ and $C_{2nh\pm x}=(2nh\pm x)c$. The response can be very well approximated by considering only few terms of the sum. For $x=h$ the response for the distributed RC line reduces to that given in [18] and is given by

$$
v_h(t)=2V_0\sum_{n=0}^{\infty}(-1)^n erfc\left(\sqrt{\frac{R_{(2n+1)h}C_{(2n+1)h}}{4t}}\right)=2V_0\sum_{n=0}^{\infty}(-1)^n erfc\left((2n+1)\sqrt{\frac{R_hC_h}{4t}}\right)
$$

The total voltage can be approximated by using only few a terms in the series given the assumption: $e^{2h\sqrt{src}}\gg 1$. For this case, C_1 and C_2 reduce to

$$
C_1=f(s);\quad C_2=\frac{f(s)}{e^{2h\sqrt{src}}}
$$

and the voltage in transform domain is

$$
V(x,s) \;=\; f(s)\left[e^{-x\sqrt{src}}+e^{-(2h-x)\sqrt{src}}\right]
$$

Substituting $f(s)=\frac{V_0}{s}$ for a step input and taking the inverse transform, we obtain

$$
v(x,t)=V_0\left[erfc\left(\sqrt{\frac{R_xC_x}{4t}}\right)+erfc\left(\sqrt{\frac{R_{2h-x}C_{2h-x}}{4t}}\right)\right] \qquad (53)
$$

where $R_x=xr$ and $C_x=xc$.

[9]By, $\mathcal{L}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}^{-1}=1-erf\left(\frac{a}{2\sqrt{t}}\right)=erfc\left(\frac{a}{2\sqrt{t}}\right)$

Using the **Boundary Condition 2** and substituting $x = 0$ in Equation (46),

$$C_1 + C_2 = f(s) = \frac{V_0}{s} \tag{47}$$

For the semi-infinite line or for only the incident diffusion component, as $x \to \infty$, the voltage on the line $V(x,s) \to 0$, which implies $C_2 = 0$, $C_1 = \frac{V_0}{s}$ and $V_I(x,s) = \frac{V_0}{s} e^{-x\sqrt{src}}$.

From the **Boundary Condition 3**, the current at the end of the transmission line is the same as the load current, i.e.,

$$I(h,s) = \frac{V_{Tot}(h,s)}{Z_L}$$

For a finite-length transmission line, the total voltage should satisfy the boundary condition at the end of the line, which is

$$I(h,s) = \frac{V_{Tot}(h,s)}{Z_L} = \lim_{x \to h} \frac{-1}{r} \frac{\partial V(x,s)}{\partial x} \tag{48}$$

Using the above condition at the end of the line, we can solve for the unknown parameters of the total voltage on the line given in Equation (46). Since for an open-ended line $I(h,s) = 0$,

$$\lim_{x \to h} \frac{-1}{r} \frac{\partial V(x,s)}{\partial x} = 0$$

i.e.,

$$-C_1 \sqrt{src}\; e^{-h\sqrt{src}} + C_2 \sqrt{src}\; e^{h\sqrt{src}} = 0$$

which yields

$$C_1 = C_2 e^{2h\sqrt{src}} \tag{49}$$

Solving for C_1 and C_2, we get

$$C_1 = \frac{f(s) e^{2h\sqrt{src}}}{1 + e^{2h\sqrt{src}}}$$
$$C_2 = \frac{f(s)}{1 + e^{2h\sqrt{src}}}$$

and the general expression for the voltage in the transform domain is

$$\begin{aligned} V(x,s) &= \frac{f(s)}{1 + e^{2h\sqrt{src}}} \left[e^{x\sqrt{src}} + e^{(2h-x)\sqrt{src}} \right] \\ &= \frac{f(s)}{1 + e^{-2h\sqrt{src}}} \left[e^{-x\sqrt{src}} + e^{-(2h-x)\sqrt{src}} \right] \end{aligned} \tag{50}$$

Appendix C: Diffusion Equation Analysis In the Transform Domain (Correction to Appendix D of Original TR-940011 April 1994)

In this Appendix, we solve the diffusion equation (45) using Laplace transform techniques. Voltage on the line in the time domain and the transform domain will be indicated by $v(x,t)$ and $V(x,s)$ respectively.

$$V(x,s) = \int_0^\infty e^{-st} v(x,t) dt$$

Applying the Laplace transform to $\frac{\partial v(x,t)}{\partial t}$ we get

$$\int_0^\infty e^{-st} \left(\frac{\partial v(x,t)}{\partial t} \right) dt = e^{-st} v(x,t)|_{t=0}^{\infty} + s \int_0^\infty e^{-st} v(x,t) dt \tag{44}$$

Since $v(x,t) = 0$ for $x > 0$ and $t = 0$ (recall **Boundary Condition 1**), $\lim_{t \to 0} e^{-st} v(x,t) = 0$. Therefore,

$$\int_0^\infty e^{-st} \left(\frac{\partial v(x,t)}{\partial t} \right) dt = s \int_0^\infty e^{-st} v(x,t) dt = sV(x,s)$$

Also,

$$\int_0^\infty e^{-st} \left(\frac{\partial^2 v(x,t)}{\partial x^2} \right) dt = \frac{\partial^2 V(x,s)}{\partial x^2}$$

Consider the diffusion equation

$$rc \frac{\partial v}{\partial t} = \frac{\partial^2 v}{n} \partial x^2 \tag{45}$$

Applying Laplace transforms to both sides yields

$$\begin{aligned} rc \int_0^\infty e^{-st} \left(\frac{\partial v(x,t)}{\partial t} \right) dt &= \int_0^\infty e^{-st} \left(\frac{\partial^2 v(x,t)}{\partial x^2} \right) dt \\ rcsV(x,s) &= \frac{\partial^2 V(x,s)}{\partial x^2} \end{aligned}$$

and solving this differential equation gives

$$V(x,s) = C_1 e^{-x\sqrt{src}} + C_2 e^{x\sqrt{src}} \tag{46}$$

Since the input voltage applied at $x = 0$ is a function of time, the transform voltage will be a function of s, i.e., $V(0,s) = f(s)$. For a step input of magnitude V_0,

$$V(0,s) = f(s) = \frac{V_0}{s}$$

Therefore, (39) reduces to

$$\frac{\partial P}{\partial \eta} + \eta P = 0. \tag{40}$$

Integrating with respect to η gives

$$\ln P = -\frac{\eta^2}{2} + C_0$$

where C_0 is a constant. This implies

$$\ln\left(\frac{\partial v}{\partial \eta}\right) + \frac{\eta^2}{2} \quad = \quad C_0 \tag{41}$$

and

$$\frac{\partial v}{\partial \eta} = e^{C_0 - \frac{\eta^2}{2}} = C_1 e^{-\frac{\eta^2}{2}}.$$

Again integrating with respect to η,

$$v(\eta) = C_1 \int_0^{\eta} e^{-\frac{\eta^2}{2}} d\eta + C_2 \tag{42}$$

Note that the error function $erf(x)$ is given by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \tag{43}$$

Appendix B: Solution of the Diffusion Equation

Here, we give the solution for the diffusion equation Equation (13):

$$rc\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \tag{35}$$

Letting $\eta = x\sqrt{\frac{rc}{2t}}$, and expressing the diffusion equation in terms of η,

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} \\ &= \frac{\partial v}{\partial \eta} \cdot -x\frac{rc^1/2}{2t^3/2} \\ &= -\frac{\eta}{2t} \cdot \frac{\partial v}{\partial \eta} \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \sqrt{\frac{rc}{2t}} \cdot \frac{\partial v}{\partial \eta} \end{aligned} \tag{37}$$

$$\begin{aligned} \Longrightarrow \frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial x}\right)\frac{\partial \eta}{\partial x} \\ &= \frac{\partial}{\partial \eta}\left(\sqrt{\frac{rc}{2t}}\frac{\partial v}{\partial \eta}\right) \cdot \sqrt{\frac{rc}{2t}} \\ &= \frac{rc}{2t}\frac{\partial^2 v}{\partial \eta^2} \end{aligned} \tag{38}$$

Substituting the above equations into Equation (35), we obtain

$$\begin{aligned} rc \cdot -\frac{\eta}{2t}\frac{\partial v}{\partial \eta} &= \frac{rc}{2t}\frac{\partial^2 v}{\partial \eta^2} \\ -\eta\frac{\partial v}{\partial \eta} &= \frac{\partial^2 v}{\partial \eta^2} \\ \frac{\partial^2 v}{\partial \eta^2} + \eta\frac{\partial v}{\partial \eta} &= 0. \end{aligned} \tag{39}$$

Equation (39) can be solved using the substitution $\frac{\partial v}{\partial \eta} = P$, so that

$$\frac{\partial^2 v}{\partial \eta^2} = \frac{\partial P}{\partial \eta}$$

Appendix A: Voltage response for open-ended distributed RC line, calculated from impulse response of [PB69].

Recall that [20] gave expressions for the transfer function (or impulse response) for a distributed RC line in the two cases of small t and large t. To enable comparison with other analyses, we extend these results by calculating the step response using Laplace transform tables [12].

For small t (i.e., $t \ll RC$), the transfer function calculated from Equation (3) is

$$H(s) = 2e^{-\sqrt{RCs}}$$

and the response is given by

$$V_2(s) = \frac{V_0}{s} \cdot H(s) \tag{32}$$

Therefore,

$$v_2(t) = 2V_0[1 - erf(\sqrt{\frac{RC}{4t}})]. \tag{33}$$

This is exactly the same as Equation (3), which was obtained by Wilnai [28].

For large t (i.e., $t \gg RC$), the transfer function obtained from Equation (4) is

$$H(s) = \frac{\pi}{\sqrt{RC}} \cdot \frac{1}{s + \frac{\pi^2}{4RC}}$$

so that the time-domain response is

$$\begin{aligned} v_2(t) &= V_0 \frac{4\sqrt{RC}}{\pi}[1 - e^{-\frac{\pi^2 t}{4RC}}] \\ &= V_0 \frac{4\sqrt{RC}}{\pi}[1 - e^{-2.467\frac{t}{RC}}]. \end{aligned} \tag{34}$$

This expression is not the same as the corresponding expression for large t given by Wilnai (Equation (3)), and one moreover needs the value of the RC constant to calculate the delay to reach a given threshold voltage. In Figure 5, we plot voltage response according to this model using $RC = 1$; higher values of RC will shift the curve to the left, implying even smaller delay times.

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shows more clearly the difference between the approximate time-domain solution of Kaufman and Garrett and our diffusion solution.

5 Summary

A survey of three decades of interconnect delay analyses reveals that the analysis of signal delay in a transmission line is traditionally performed starting with a lossless LC representation and a *wave equation* for the system response; the solution is obtained in the transform domain via 2-port parameters. In this paper, we begin with a distributed RC line model of the interconnect, which yields a *diffusion equation* for the voltage response. We have given a new analytic solution of this equation incorporating appropriate boundary conditions, via a pure time-domain approach. Using appropriate analysis of reflections, we obtain the classical result for voltage response in a finite-length open-ended line, i.e., matching the solution derived from transform techniques [28, 23, 18]. We emphasize that of over ten approaches in the literature for this case alone, ours is the first purely time-domain solution (e.g., contrast with [17]). Our method can explicitly incorporate time-of-flight and reflection coefficients from arbitrary source and load discontinuities. Since our computation of total response is based on summing various individual components of diffusion, each of which can be evaluated in the time domain, the total voltage can be easily estimated up to the required accuracy.

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Methods	Delay Time For		
	50 %	**63.2 %**	**90 %**
Simple Lumped Model	$0.69RC$	$1.0RC$	$2.30RC$
Wilnai's Model	$0.39RC$	$0.515RC$	$1.03RC$
Kaufman's or Sakurai's Model	$0.38RC$	$0.5RC$	$1.03RC$
Diffusion Equation Model	$0.38RC$	$0.5RC$	$1.03RC$

Table 2: Delay times for various thresholds and models.

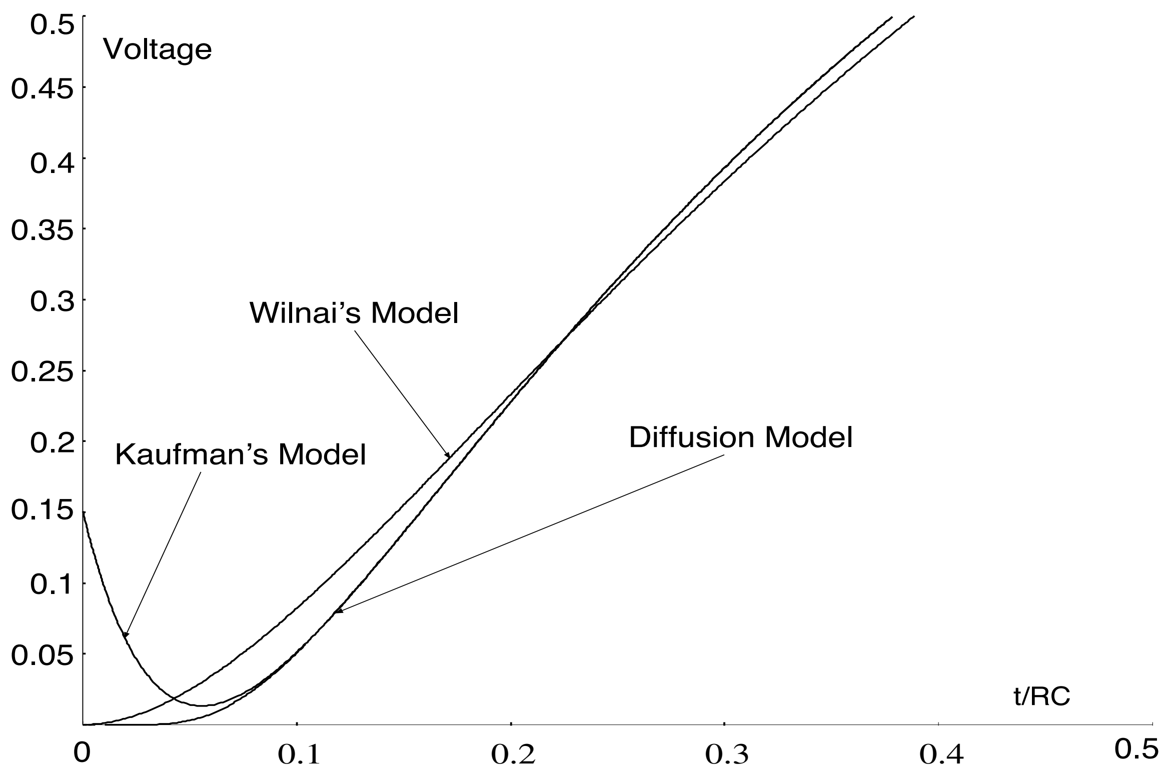

Figure 6: Kaufman's model with non-monotone response compared to Wilnai's and our diffusion response.

We now give delay calculations for the open-ended distributed RC line with an ideal source. Recall that the solution with an ideal step input is obtained in Equation (29). Using error function tables, we can easily calculate the time for a signal to reach a given threshold voltage at distance x (on the line) from the input terminal. For example, the diffusion equation solution implies a 63% delay time of $t_x(63\%) = 0.5 \cdot R_x C_x$. For comparison, Table 1 gives the step input response for an open-ended distributed RC line for each of the various approaches. Figure 5 shows the delay times for the diffusion equation model as compared to previous approaches. Figure 6

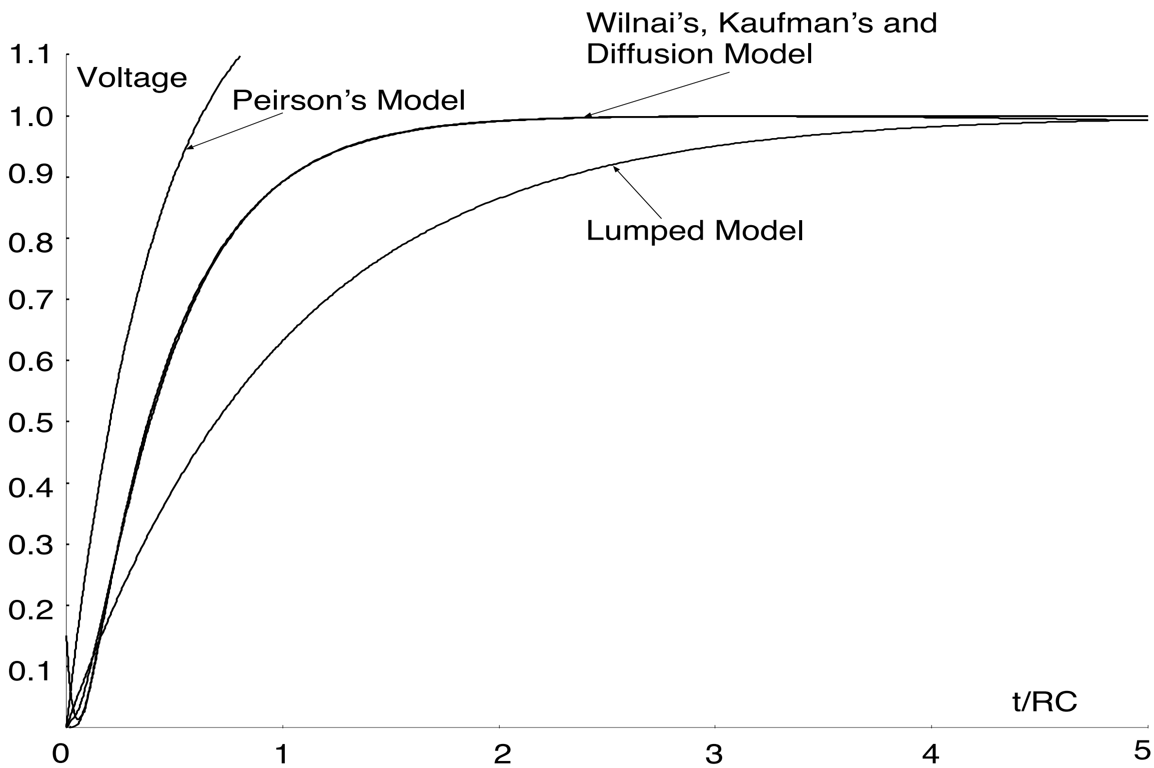

Figure 5: Unit step response for lumped RC model and various distributed RC models. Note that Wilnai's and Kaufman's models are not identical: Kaufman's ($\equiv$ Sakurai's) model gives a non-monotone response. Also note that the response in Peirson et al.'s model is dependent on the RC constant; we have plotted the response for $RC = 1$.

For $x = h$ the response for the distributed RC line reduces to that given in [18] and is given by

$$v_{Tot}(h,t) = 2V_0 \sum_{n=0}^{\infty}(-1)^n erfc\left(\sqrt{\frac{R_{(2n+1)h}C_{(2n+1)h}}{4t}}\right) = 2V_0 \sum_{n=0}^{\infty}(-1)^n erfc\left((2n+1)\sqrt{\frac{R_hC_h}{4t}}\right) \tag{30}$$

Finally, we show that the expression for total voltage satisfies the boundary condition at the end of the line (24).

$$\begin{aligned} i(h,t) &= \lim_{x\to h} \frac{-1}{r}\frac{\partial v_{Tot}(x,t)}{\partial x} \\ &= \lim_{x\to h} V_0\sqrt{\frac{rc}{t\pi}}e^{\frac{rcx^2}{4t}} + V_0\sqrt{\frac{rc}{t\pi}}\sum_{n=1}^{\infty}\left[(-1)^n e^{\frac{rc(2nh+x)^2}{4t}} + (-1)^n e^{\frac{rc(2nh-x)^2}{4t}}\right] \\ &= V_0\sqrt{\frac{rc}{t\pi}}e^{\frac{rch^2}{4t}} + V_0\sqrt{\frac{rc}{t\pi}}\sum_{n=1}^{\infty}\left[(-1)^n e^{\frac{rch^2(2n+1)^2}{4t}} + (-1)^n e^{\frac{rch^2(2n-1)^2}{4t}}\right] = 0 \end{aligned} \tag{31}$$

4.2 Threshold Delay Calculations Using Diffusion Equation Model

The total voltage can be computed as a sum of the incident diffusion and the reflected diffusions. The first reflected voltage is

$$V_{R_1}(x,s) = \Gamma_L(s)V_I(2h-x,s) = V_I(2h-x,s)$$

and the corresponding time-domain response is

$$V_{R_1}(x,t) = V_I(2h-x,t) = V_0 erfc\left(\sqrt{\frac{R_{2h-x}C_{2h-x}}{4t}}\right)$$

Similarly, the response of the second reflection is given by

$$V_{R_2}(x,s) = \Gamma_L(s)\Gamma_S(s)V_I(2h-x,s) = -V_I(2h+x,s)$$

from which

$$V_{R_2}(x,t) = -V_I(2h+x,t) = -V_0 erfc\left(\sqrt{\frac{R_{2h+x}C_{2h+x}}{4t}}\right)$$

Extending, the i^{th} reflection for even $i = 2n$ in the transform domain is

$$\begin{aligned} V_{R_i}(x,s) &= \Gamma_L^{\frac{i}{2}}(s)\Gamma_S^{\frac{i}{2}}(s)V_I(ih+x,s) = (-1)^{\frac{i}{2}}V_I(ih+x,s) \\ &= (-1)^n V_I(2nh+x,s) \end{aligned} \tag{26}$$

and the i^{th} reflection for odd $i = 2n-1$ is

$$\begin{aligned} V_{R_i}(x,s) &= \Gamma_L^{\frac{i+1}{2}}(s)\Gamma_S^{\frac{i-1}{2}}V_I(h(i+1)-x,s) = (-1)^{\frac{i-1}{2}}V_I(h(i+1)-x,s) \\ &= (-1)^{n-1}V_I(2nh-x,s) \end{aligned} \tag{27}$$

and the respective time-domain responses are

$$v_{R_n}(x,t) = \begin{cases} (-1)^n v_I(2nh+x,t) = (-1)^n erfc\left(\sqrt{\frac{R_{2nh+x}C_{2nh+x}}{4t}}\right) & \text{for even } n \\ (-1)^{n-1}v_I(2nh-x,t) = (-1)^{n-1} erfc\left(\sqrt{\frac{R_{2nh-x}C_{2nh-x}}{4t}}\right) & \text{for odd } n \end{cases} \tag{28}$$

Therefore, the total voltage is

$$\begin{aligned} v_{tot}(x,t) &= v_I(x,t) + \sum_{n=1}^{\infty}\left[(-1)^n v_I(2nh+x)(x,t) + (-1)^{n-1}v_I(2nh-x,t)\right] \\ &= V_0 erfc\left(\sqrt{\frac{R_xC_x}{4t}}\right) \\ &\quad + V_0\sum_{n=1}^{\infty}\left[(-1)^n erfc\left(\sqrt{\frac{R_{2nh+x}C_{2nh+x}}{4t}}\right) + (-1)^{n-1} erfc\left(\sqrt{\frac{R_{2nh-x}C_{2nh-x}}{4t}}\right)\right] \end{aligned} \tag{29}$$

$\Gamma_L(s)V_I(2h-x,s)$. We thus obtain

$$\begin{aligned} V_{Tot}(x,s) &= V_I(x,s) + V_{R_1}(x,s) \\ &= V_I(x,s) + \Gamma_L(s)V_I(2h-x,s) \end{aligned}$$

and the time-domain response of the first reflection at load is

$$v_{R_1}(x,t) = \Gamma_L(t) \otimes v_I(2h-x,t) = \int_{\tau=0}^{t} \Gamma_L(t-\tau)v_I(2h-x,\tau)\, d\tau.$$

Therefore, the total voltage on the line is

$$v_{Tot}(x,t) = v_I(x,t) + \int_{\tau=0}^{t} \Gamma_L(t-\tau)v_I(2h-x,\tau)\, d\tau.$$

In the special case of an open-ended line, $Z_L = \infty$ and $\Gamma_L(s) = 1$, $\Gamma_L(t) = \delta(t)$, so that

$$\begin{aligned} v_{Tot}(x,t) &= v_I(x,t) + \int_{\tau=0}^{t} \delta(t-\tau)v_I(2h-x,\tau)\, d\tau \\ &= v_I(x,t) + v_I(2h-x,t) \\ &= V_0 erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) + V_0 erfc\left(\sqrt{\frac{R_{2h-x}C_{2h-x}}{4t}}\right) \end{aligned}$$

and the total voltage at the end of the line is

$$v_{Tot}(h,t) = 2V_0 erfc\left(\sqrt{\frac{R_h C_h}{4t}}\right)$$

Again, applying the boundary condition (24) at the end of the line, we get[8]

$$\begin{aligned} i(h,t) &= \lim_{x\to h} \frac{-1}{r}\frac{\partial v_{Tot}(x,t)}{\partial x} \\ &= \lim_{x\to h} \left[V_0\sqrt{\frac{rc}{t\pi}}e^{\frac{-rcx^2}{4t}} - V_0\sqrt{\frac{rc}{t\pi}}e^{\frac{-rc(2h-x)^2}{4t}}\right] = 0 \end{aligned}$$

In practice, the open-ended approximation of an interconnect is often used since the input impedance of MOS devices is typically high compared to the characteristic impedance of the line.

Case 4: Finite-length, open-ended RC transmission line with ideal source.

With $Z_L = \infty$ (i.e., $\Gamma_L(s) = 1$ and $\Gamma_L(t) = \delta(t)$), and an ideal source ($Z_S = 0$, $\Gamma_S = -1$ and $\Gamma_S(t) = \delta(t)$), we obtain a case that has received much previous attention, as noted in Section 2.

[8] By $\frac{\partial v_{Tot}(x,t)}{\partial x} = \frac{\partial v_{Tot}(x,t)}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$

Case 2: Finite-length RC transmission line with perfectly matched load [20].

If $Z_L = Z_0$, there is no reflection at the load (all V_{R_i} are zero), i.e.,

$$v_{Tot}(x,t) = v_I(x,t)$$

The input impedance of the line from the source side for the first incident diffusion is same as the characteristic impedance of the line (see page 234 of [3]). So the voltage at the front end of the line in time-domain is[7]

$$v_1(t) = \mathcal{L}\left\{(\frac{Z_0}{Z_0+Z_S})\frac{V_0}{s}\right\}^{-1} = \mathcal{L}\left\{\frac{1-\Gamma_S}{2}\frac{V_0}{s}\right\}^{-1}$$

and α_{rise} is calculated from the time-domain expression of the voltage $v_1(t)$ at the left port of the line as discussed in Section 3.2. Therefore, the total voltage is given by

$$v_{Tot}(x,t) = \kappa\alpha_{rise}V_0(1 - erf(\frac{\eta}{\sqrt{2}})) \tag{25}$$

and for a step input reduces to

$$v_{Tot}(x,t) = V_0(1 - erf(\frac{\eta}{\sqrt{2}})) = V_0 erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right)$$

It can be easily verified that the total voltage satisfies the boundary condition (24) at the end of the line. (Note that Peirson et al. [20] have used the assumption of a matched termination to derive the impulse response at an arbitrary location x on the distributed RC line. This turns out to be the same as the equation for the incident wave (20) as there are no other reflected voltages.)

Case 3: Finite-length RC transmission line with perfectly matched source and general load.

A finite-length transmission line with source impedance $Z_S = Z_0$ (i.e., $\Gamma_S = 0$) will have only the single initial reflection at the load. For a general load impedance Z_L, we have $V_{R_1}(x,s) =$

[7] The input impedance in terms of $ABCD$ parameters is [9]

$$Z_{in} = \frac{cosh(\theta h)\cdot Z_L + Z_0 \cdot sinh(\theta h)}{Y_0 \cdot Z_L \cdot sinh(\theta h) + cosh(\theta h)}$$

where the propagation constant $\theta h = \sqrt{sRC}$, and the characteristic impedance $Z_0 = \sqrt{\frac{R}{sC}}$. From the input impedance expression of a 2-port interconnect line a perfectly matched load implies $Z_{in} = Z_0$, $V_1(s) = (\frac{Z_{in}}{Z_{in}+Z_S})\frac{V_0}{s} = (\frac{Z_0}{Z_0+Z_S})\frac{V_0}{s}$.

and in general the i^{th} reflection gives

$$V_{R_i}(x,s) = \begin{cases} \Gamma_L^{\frac{i}{2}}(s)\Gamma_S^{\frac{i}{2}}(s)V_I(ih+x) & \text{for } i \text{ even} \\ \Gamma_L^{\frac{i+1}{2}}(s)\Gamma_S^{\frac{i-1}{2}}V_I(h(i+1)-x) & \text{for } i \text{ odd} \end{cases} \tag{22}$$

Substituting $i = 2n$ for even i, and $i = 2n-1$ for odd i, we get

$$\begin{aligned} V_{Tot}(x,s) &= V_I(x,s) + \sum_{i=1}^{\infty} V_{R_i}(x,s) \\ &= V_I(x,s) + \sum_{n=1}^{\infty}\left(\Gamma_L^n(s)\Gamma_S^{n-1}V_I(2nh-x,s) + \Gamma_L^n(s)\Gamma_S^n(s)V_I(2nh+x,s)\right) \end{aligned}$$

and the time-domain response is

$$\begin{aligned} v_{Tot}(x,t) &= v_I(x,t) + \sum_{n=1}^{\infty}\left[a_n(t)\otimes v_I(2nh-x,t) + b_n(t)\otimes v_I(2nh+x,t)\right] \\ &= v_I(x,t) + \sum_{n=1}^{\infty}\left[\int_{\tau=0}^{t} a_n(t-\tau)v_I(2nh-x,\tau)\,d\tau + \int_{\tau=0}^{t} b_n(t-\tau)v_I(2nh+x,\tau)\,d\tau\right] \end{aligned} \tag{23}$$

where $a_n(t)$ and $b_n(t)$ represent odd and even n^{th} reflection coefficient values. The total voltage at the end of the transmission line of length h is

$$v_{Tot}(h,t) = v_I(h,t) + \sum_{n=1}^{\infty}\left[\int_{\tau=0}^{t} a_n(t-\tau)v_I(h(2n-1),\tau)\,d\tau + \int_{\tau=0}^{t} b_n(t-\tau)v_I(h(2n+1),\tau)\,d\tau\right]$$

For a finite-length transmission line, the total voltage should satisfy the boundary condition at the end of the line, which is

$$i(h,t) = \lim_{x\to h}\frac{-1}{r}\frac{\partial v_{Tot}(x,t)}{\partial x} \tag{24}$$

Also, the current at the end of the line is the same as the load current, i.e.,

$$I(h,s) = \frac{V_{Tot}(h,s)}{Z_L}$$

Hence, the time-domain current can be obtained for any specific case by taking the inverse Laplace transform of $I(h,s)$. We may now consider cases of source and load impedances which have received particular attention throughout the literature.

Case 1: Semi-infinite RC transmission line.

Since the line is semi-infinite, there are no reflections at the load, $V_{R_i} = 0$. For a step input with no source impedance the time-domain response is

$$v_{Tot}(x,t) = v_I(x,t) = V_0 erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right)$$

General Case: Finite-length RC transmission line with general source and load impedances Z_S and Z_L.

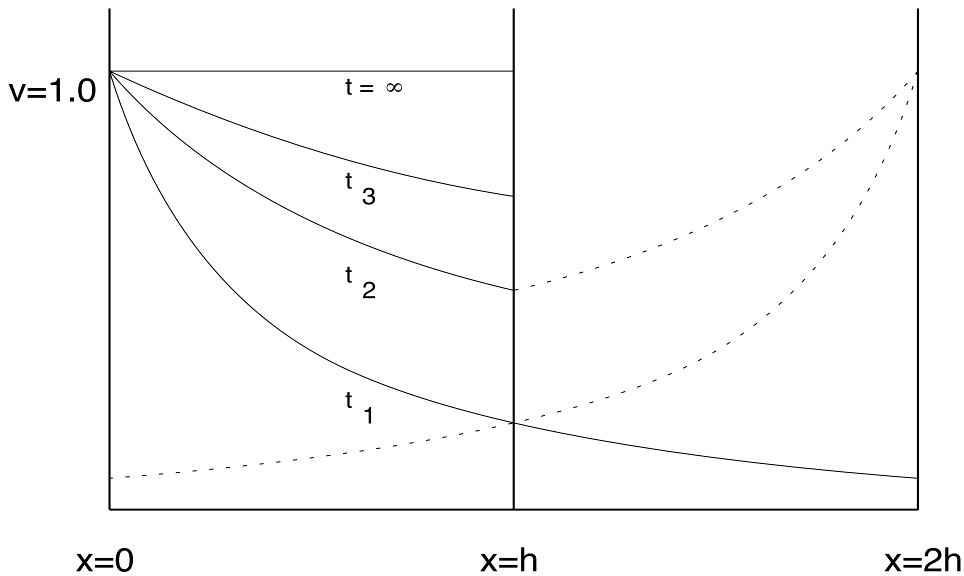

Figure 4: The evolution of the voltage profile for a finite length distributed RC line. The dotted line indicates the voltage due to the first reflection at $x = h$, which can be also be obtained from the Method of Images (or Reflections) [Kev90].

The reflection coefficient at the source in transform domain is $\Gamma_S(s) = \frac{Z_S - Z_0}{Z_S + Z_0}$, and the reflection coefficient at the load is $\Gamma_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0}$. The voltage at the position x (Figure 3) due to the first reflection at the load can be calculated by considering the incident wave and shifting in position by $h + h - x = 2h - x$ and is given by

$$V_{R_1}(x, s) = \Gamma_L(s) V_I(2h - x, s).$$

The corresponding time-domain expression is

$$v_{R_1}(x, t) = \int_{\tau=0}^{t} \Gamma_L(t - \tau) v_I(2h - x, \tau)\, d\tau,$$

i.e., the first reflected voltage travels a distance h to the end of the line before reflection and then $h - x$ to reach the desired location. Another explanation for the reflection voltages is by the symmetry argument in the Method of Images (or Reflections) [14] to satisfy the boundary condition at the end of the line $x = h$, as shown in Figure 4.

Similarly, the second reflection at the source yields

$$V_{R_2}(x, s) = \Gamma_S(s) \Gamma_L(s) V_I(2h + x, s)$$

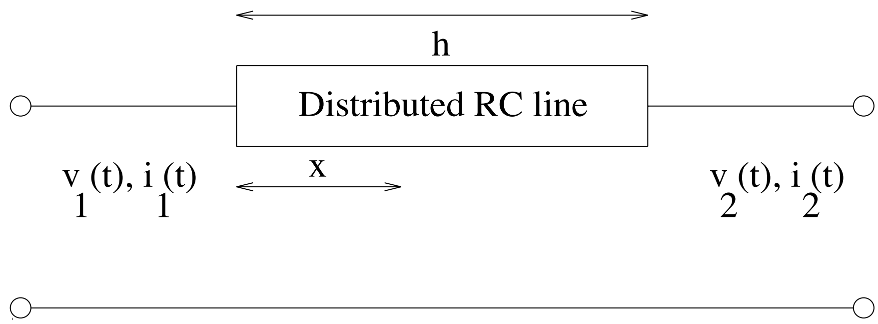

Figure 3: A distributed RC line of length h, and position x along the line.

The solution to the diffusion equation corresponds to the "incident" propagation of voltage due to the first diffusion for a step input:

$$v_I(x,t) = v_I(\eta) = V_0[1 - erf(\frac{\eta}{\sqrt{2}})] = V_0 erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) \tag{20}$$

where $R_x = xr$, $C_x = xc$ and r, c are resistance and capacitance per unit length.[4] Recall that the total voltage on the line (Figure 3) is given by the summation on incident component plus reflected components. The reflections are due to discontinuities, e.g., at the source (S) and load (L). In other words, the time-domain expansion for total voltage is

$$v_{Tot}(x,t) = v_I(x,t) + \sum_{i=1}^{\infty} v_{R_i}(x,t) \tag{21}$$

where $v_I(x,t) \equiv$ voltage corresponding to the incident diffusion and $v_{R_i}(x,t) \equiv$ voltage corresponding to the i^{th} reflection.[5] At any given time, the expression for any individual $v_{R_i}(x,t)$ will be of the same form as the expression for the incident voltage $v_I(x,t)$, but with a displacement in the position variable and also with a different magnitude (according to the reflection coefficients). We use this observation to calculate the total voltage on the line for general source and load impedances. In general $v_{R_i}(x,t)$ can be calculated through convolution of the reflected diffusion taking into account position displacement, with the reflection coefficients $\Gamma_S(t)$ or $\Gamma_L(t)$.[6] Alternatively, one may also take the inverse Laplace transform of $v_{R_i}(x,s)$. (Note that a different, incorrect analysis of reflections was given in [16]; this led to some recent confusion (e.g., [21]) which the present discussion alleviates.)

[4] From [12] the Laplace transform of Equation (20) is $V_I(x,s) = \frac{V_0}{s} e^{-\sqrt{R_x C_x s}}$.

[5] Similarly, the total voltage in transform domain is $V_{Tot}(x,s) = V_I(x,s) + \sum_{i=1}^{\infty} V_{R_i}(x,s)$.

[6] Note that $\Gamma_S(t) = \mathcal{L}\{\Gamma_S(s)\}^{-1}$.

To achieve the case of an ideal step input, the boundary condition at $x = 0$ should be evaluated for t_{rise} tending to zero and α_{rise} tending to 1, i.e., we let $t_{rise} = \varepsilon$ with $\varepsilon \to 0$. Then, the error function argument in the expression for κ will tend to zero, since ϵ and ε both tend to zero with the numerator of higher degree than the denominator, i.e.,

$$\lim_{\epsilon \to 0, \varepsilon \to 0} \frac{\epsilon}{2}\sqrt{\frac{rc}{\varepsilon}} = 0.$$

Therefore, $\kappa = 1$ and the diffusion equation solution reduces to

$$v(\eta) = V_0[1 - erf(\frac{\eta}{\sqrt{2}})]. \tag{18}$$

Observe that the same result is obtained when there is an ideal source at the input. Here, the voltage at the front end of the line is $v_1(0,t) = V_0 u(t)$ and $V_1(s) = \frac{V_0}{s}$, i.e., voltage is constant at $x = 0$ for all $t > 0$ and equal to V_0. Using this condition in Equation (13) yields

$$V_0 = C_1 \int_0^{\eta = x\sqrt{\frac{rc}{2t}} = 0} e^{-\frac{x^2}{2}} dx + C_2$$

from which $C_2 = V_0$ and

$$v(\eta) = V_0[1 - erf(\frac{\eta}{\sqrt{2}})]. \tag{19}$$

This result is the same as Equation (18), as we expect.

Two comments are is in order: (i) the two boundary conditions we use are discontinuous (at $x = 0$, $t = 0$), but this discontinuity smooths immediately and the solution is still valid, and (ii) in general, the voltage at any point on the line is expressed as a summation of error functions due to the reflections at source and load discontinuities.

4 Applying the Diffusion Equation Analysis

We close our development by extending the basic result of Equation (18) in two ways: (i) introducing an analysis of reflections which allows calculation of the total voltage response for various configurations of the distributed RC line, and (ii) incorporating non-zero time of flight into the analysis.

4.1 Analysis of Reflections

value, t_{rise}. Notice that the voltage at $x = 0$ depends on the source impedance, Z_S, and the characteristic impedance of the line, Z_0, since this structure acts as a voltage divider. In the transform domain, the voltage at $x = 0$ is given by

$$V_1(s) = (\frac{Z_0}{Z_0 + Z_S})\frac{V_0}{s} \tag{15}$$

where $V_1(s)$ is the voltage of the first incident diffusion at the front end of the line ($x = 0$).

At a given rise-time ($t = t_{rise}$), the voltage $v_1(0, t_{rise})$ at the front end of the transmission line can be obtained from the time domain representation of Equation (15). Let $\alpha_{rise}V_0$ be this voltage at the rise-time, i.e., $V_1(0, t_{rise}) = \alpha_{rise}V_0$, where $0 < \alpha_{rise} \leq 1$.

Substituting into Equation (13) and evaluating at $x = \epsilon$, with ϵ tending to 0, we obtain

$$\alpha_{rise}V_0 = C_1 \int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}} e^{(-\frac{\eta^2}{2})} d\eta + C_2 \tag{16}$$

3.3 Solution of the Diffusion Equation

We use (14) and (16) to solve for C_1 and C_2 :

$$\alpha_{rise}V_0 = C_1 \int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}} e^{(-\frac{x^2}{2})} dx - C_1\sqrt{\frac{\pi}{2}}$$

yields

$$C_1 = \alpha_{rise}V_0 \cdot \frac{1}{[\int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}} e^{-\frac{x^2}{2}} dx - \sqrt{\frac{\pi}{2}}]}$$

from which

$$\begin{aligned}
v(\eta) &= \alpha_{rise}V_0 \cdot \frac{1}{[\int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}} e^{-\frac{x^2}{2}} dx - \sqrt{\frac{\pi}{2}}]} \cdot [\int_0^{\eta} e^{-\frac{x^2}{2}} dx - \sqrt{\frac{\pi}{2}}] \\
&= \alpha_{rise}V_0 \cdot \frac{1}{[1 - \sqrt{\frac{2}{\pi}} \int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}} e^{-\frac{x^2}{2}} dx]} \cdot [1 - \sqrt{\frac{2}{\pi}} \int_0^{\eta} e^{-\frac{x^2}{2}} dx] \\
&= \alpha_{rise}V_0 \cdot \frac{1}{[1 - erf(\frac{\epsilon}{2}\sqrt{\frac{rc}{t_{rise}}})]} \cdot [1 - erf(\frac{\eta}{\sqrt{2}})] \\
&= \kappa\alpha_{rise}V_0[1 - erf(\frac{\eta}{\sqrt{2}})] \qquad (17)
\end{aligned}$$

where

$$\kappa = \frac{1}{[1 - erf(\frac{\epsilon}{2}\sqrt{\frac{rc}{t_{rise}}})]}.$$

and then substituting, we obtain the **diffusion equation**

$$\frac{\partial}{\partial x}[-\frac{1}{r}\frac{\partial v}{\partial x}] = -c\frac{dv}{dt}$$

which implies

$$rc\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}. \tag{11}$$

One can similarly derive the diffusion equation for current on the line:

$$rc\frac{\partial i}{\partial t} = \frac{\partial^2 i}{\partial x^2}. \tag{12}$$

The solution of Equation (11) can be obtained by restricting it to the set of solutions of the form $v(\frac{x}{\sqrt{t}})$ using the substitution variable $\eta = x\sqrt{\frac{rc}{2t}}$ (see Appendix B). This is the appropriate substitution for a parabolic equation. We obtain the general solution

$$v(\eta) = C_1 \int_0^\eta e^{-\frac{\eta^2}{2}} d\eta + C_2 \tag{13}$$

One can also obtain this solution directly from the heat kernel for the diffusion equation [7]. This should be contrasted with the solution of [17], which must assume the separable form for $v(x,t)$. Of course, specific solutions of interest will depend on the boundary conditions that apply; in particular, we are interested in the well-studied case of the open-ended distributed RC line.

3.2 Boundary Conditions

We now derive the two boundary conditions necessary to solve Equation (13) for the distributed RC line.

Boundary Condition 1: At $t = 0$, the line is quiet and $v(x,t) = 0$ for all x, i.e.,

$$C_1\sqrt{\frac{\pi}{2}} + C_2 = 0.$$

Therefore,

$$C_2 = -C_1\sqrt{\frac{\pi}{2}} \tag{14}$$

Note that this boundary condition applies to every new wave that is born due to reflection.

Boundary Condition 2: The second boundary condition is obtained from the structure of the input applied at the front end of the transmission line ($x = 0$) and in terms of the rise-time

Method	Accuracy/ Regime	Time-Domain Voltage Response
Simple Lumped Model	Approximate	$V_0(1 - e^{-\frac{t}{RC}})$
Wilnai's 2-port Model	Small t Large t	$2V_0(1 - erf(\sqrt{\frac{RC}{4t}}))$ $V_0(1 - 1.366e^{-2.5359\frac{t}{RC}} + 0.366e^{-9.4641\frac{t}{RC}})$
Mattes's 2-port Model	Exact	$2V_0\sum_{n=1}^{\infty}(-1)^{n-1}\left(1 - erf(\frac{2n-1}{2}\sqrt{\frac{RC}{t}})\right)$
Peirson's 2-port Model*	Small t Large t	$2V_0(1 - erf(\sqrt{\frac{RC}{4t}}))$ $V_0 1.273\sqrt{RC}(1 - e^{-2.467\frac{t}{RC}})$
Sakurai's 2-port Model	Exact but approximated to three terms	$V_0(1 - 1.273e^{-2.467\frac{t}{RC}} + 0.424e^{-22.206\frac{t}{RC}})$
Kaufman's Diffusion Equation Model	Heuristic derivation approximated to three terms	$V_0(1 - 1.273e^{-2.467\frac{t}{RC}} + 0.424e^{-22.206\frac{t}{RC}})$
Antinone/Brown's 2-port model	Approximate	$V_0(1 - 1.172e^{-2.467\frac{t}{RC}} + 0.195e^{-22.206\frac{t}{RC}} - 0.023e^{-61.685\frac{t}{RC}})$

Table 1: Voltage response of an open-ended distributed RC line under a step input excitation of magnitude V_0. $(*)$ Response calculated from the transfer function in [PB69].

$$v(x,t) = r(x)\Delta x i(x+\Delta x, t) + v(x+\Delta x, t)$$

where $r(x)$ and $c(x)$ are resistance and capacitance per unit length. As $\Delta x \rightarrow 0$ the above equations reduce to[3]

$$\frac{\partial i(x,t)}{\partial x} = -c(x)\frac{\partial v(x,t)}{\partial t} \tag{9}$$

$$\frac{\partial v(x,t)}{\partial x} = -r(x)i(x,t) \tag{10}$$

Assuming a uniform wire, whereby $r(x) = r$ and $c(x) = c$ are constants independent of x,

[3] Equation (9) and Equation (10) are often referred to as the *Telegrapher's* equations. These may also be derived directly from Maxwell's equations; see, e.g., [9].

2.2.2 A Previous Time-Domain Analysis

Finally, a solution which uses time-domain analysis is that of Kaufman and Garrett [17], who formulate a distributed RC model for interconnect and derive a diffusion equation for voltage on the line. However, to obtain the transient response to a step input, [17] makes the simplifying assumption $v(x,t) = f(x) \cdot g(t)$, namely, separability of the voltage response into separate functions of position and time; this leads to a complicated and special-case solution

$$v_2(t) = 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-((2n+1)\pi/2)^2 t/RC} \tag{7}$$

Considering only the first few terms of the series yields

$$v_2(t) \approx (1 - 1.273 e^{-2.467\frac{t}{RC}} + 0.424 e^{-22.206\frac{t}{RC}}) \tag{8}$$

This expression is different from that of Wilnai or Peirson, but is identical[2] to that given by Sakurai (Equation 5). The book of Ghausi and Kelly [11] gives an analysis using the same separability assumption. As we shall see below, the separability assumption leads to an error in the response. These previous delay approximations are summarized in Table 1 below.

3 The Diffusion Equation Analysis

3.1 Obtaining the Diffusion Equation from the Distributed RC Model

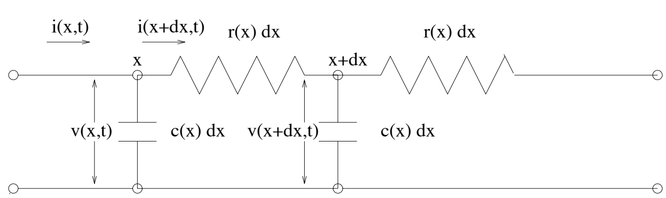

Figure 2: Lumped approximation for Δx in a distributed RC line.

The diffusion equation for voltage in a distributed RC line can be derived from first principles as follows (see [17]). Consider a lumped approximation for Δx of the line, as shown in Figure 2. By applying simple nodal equations at the nodes x and $x + \Delta x$, we obtain

$$i(x,t) \quad = \quad c(x)\Delta x \frac{\partial v(x,t)}{\partial t} + i(x+\Delta x, t)$$

[2] In the taxonomy that we present below, the result of Kaufman and Garrett is characterized as "approximate" because their method must assume the special form of the diffusion equation solution. On the other hand, Sakurai's result is "exact" with respect to the distributed RC 2-port analysis.

Sakurai [23] also uses the 2-port model and obtains the voltage response as

$$v_2(t) = V_0(1 - 1.273e^{-2.467\frac{t}{RC}} + 0.424e^{-22.206\frac{t}{RC}}) \tag{5}$$

These works, particularly [28], have had great influence on the literature. For example, Saraswat and Mohammadi [24] use the results of [17, 28] to obtain their rise time estimates. Bakoglu and Meindl [4] also cite Wilnai's derivation, and write: "Under step-voltage excitation, the times (T) required for the output voltage of distributed and lumped RC networks to rise from 0 to 90 percent of their final values are $1.0RC$ and $2.3RC$, respectively." ([4], p. 904). The authors of [4] go on to state a "very good approximation for delay":

$$\begin{aligned} T &= 1.0R_{int}C_{int} + 2.3(R_{tr}C_{int} + R_{tr}C_L + R_{int}C_L) \\ &\approx (2.3R_{tr} + R_{int})C_{int} \end{aligned} \tag{6}$$

(R_{int} and C_{int} are respectively the interconnect resistance and capacitance, R_{tr} is the output resistance of the driving transistor and C_L is the load capacitance). This last expression (6) has been frequently invoked in the literature (see [1], [25] or the book [3]).

Interestingly, the voltage response for a step input using the 2-port model has been rederived many times in the literature. For example, Antinone and Brown [2] express $cosh(\sqrt{sRC})$ as an infinite product series and then consider only the first three terms of the product expansion. This is not a good approximation because the coefficients of s and s^2 are not exact, and depend heavily on the number of terms used in the product expansion. Mey [19] noted the crudeness of this approximation and proposed an infinite partial fraction expansion, thus obtaining the same solution as Sakurai. Ghausi and Kelly [11] are yet another group who earlier published the identical analysis.

The common feature of all these works is that they use the 2-port transfer matrix of the distributed RC line to obtain their respective time-domain estimates of the transient response. The 2-port parameters for the distributed RC line are obtained from the solution of the *wave equation* (1) for v and i (see, e.g., [9]). But as we discuss in Section 3 below, voltage or current in a pure distributed RC line obeys a *diffusion equation.*

with R and C respectively denoting the lumped values of the line resistance and capacitance. Equation (2) is the basis of a number of analyses which are derived from the 2-port model. Using $\cosh x = \frac{e^x+e^{-x}}{2}$ and making the approximation $\cosh x \approx \frac{e^x}{2}$ for $Re\sqrt{sRC} \gg 1$ (i.e., the high-frequency leading edge of the step input), Wilnai obtains the approximate time-domain response

$$v_2(t) \approx \begin{cases} 2V_0[1 - erf(\sqrt{\frac{RC}{4t}})] & t \ll RC \\ V_0[1 - 1.366e^{-\frac{2.5359t}{RC}} + 0.366e^{-\frac{9.4641t}{RC}}] & t \gg RC \end{cases}$$

for the cases $t \ll RC$ and $t \gg RC$, respectively.[1] Using Equation (3), Wilnai obtains a value of $1.02RC$ for the 90% delay time and a value of $0.37RC$ for the 50% delay time. By writing

$$\begin{aligned} \frac{1}{s\cosh\sqrt{sRC}} &= \frac{1}{s} \cdot \frac{2}{e^{\sqrt{sRC}} + e^{-\sqrt{sRC}}} \\ &= \frac{1}{s} \cdot \frac{2}{e^{\sqrt{sRC}}(1 + e^{-2\sqrt{sRC}})} \end{aligned}$$

and using $\frac{1}{1+e^{-2\sqrt{sRC}}} = \sum_{n=0}^{\infty}(-e^{-2\sqrt{sRC}})^n$, Mattes [18] has recently obtained a more precise solution of Equation (2) which yields estimates of $1.06RC$ for the 90% delay time, and $0.37RC$ for the 50% delay time.

Peirson and Bertnolli [20] have also calculated the transfer function of an open-ended distributed RC line; by using reciprocal time domain analysis they find approximate time domain expressions for the transfer function (or Laplace transform of the impulse response), namely,

$$h(t) \approx [\frac{\sqrt{RC}}{\sqrt{\pi}t^{\frac{3}{2}}}]e^{-\frac{RC}{4t}} \quad \text{for small } t \quad (t \ll RC) \tag{3}$$

$$h(t) \approx [\frac{\pi}{\sqrt{RC}}]e^{-\frac{\pi^2 t}{4RC}} \quad \text{for large } t \quad (t \gg RC) \tag{4}$$

From these expressions, we may easily derive the system response in each case using Laplace transform tables (see Appendix A for details):

$$v_2(t) \approx \begin{cases} 2V_0[1 - erf(\sqrt{\frac{RC}{4t}}) & \text{for small } t \quad (t \ll RC) \\ V_0\frac{4\sqrt{RC}}{\pi}[1 - e^{-2.467\frac{t}{RC}}] & \text{for large } t \quad (t \gg RC) \end{cases}$$

[1]The *error function*, $erf(x)$, is given by $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dt$, and the complement function to the error function, $erfc(x)$, is given by $erfc(x) = 1 - erf(x)$.

model, the delay to reach the 63% voltage threshold is $1.0RC$.

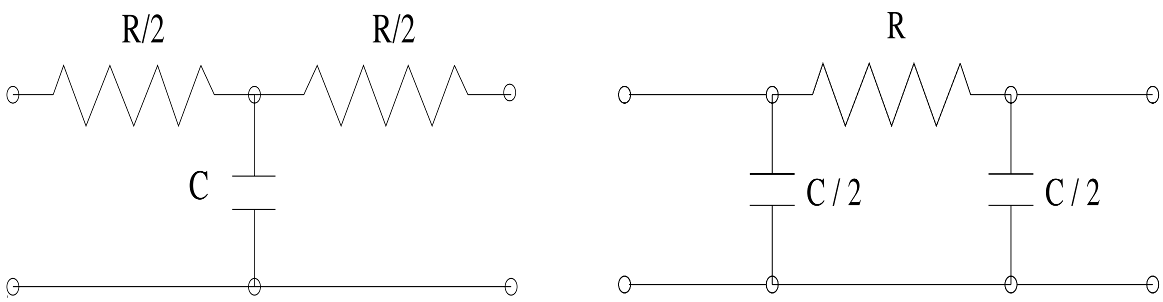

Figure 1: **T** and **Π** elements used in modeling a distributed RC line.

Many works (e.g., [26, 27]) model a distributed RC interconnect using a simple **T** or **Π** configuration, which gives a first-order "lumped-distributed" model or a single-pole response. The transfer function (or Laplace transform of the impulse response) for a **T** or **Π** configuration (Figure 1) is given by

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1 + s\frac{RC}{2}}$$

This results in a lumped circuit element with time constant

$$T = \frac{RC}{2}.$$

2.2 Distributed Models

2.2.1 Transient Response Using Laplace Transform

The $ABCD$ parameters of a distributed RC transmission line are [9]:

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta h) & Z_0 \sinh(\theta h) \\ \frac{1}{Z_0}\sinh(\theta h) & \cosh(\theta h) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix}$$

where $\theta h = \sqrt{sRC} = \sqrt{j\omega RC}$, and V_1 and V_2 respectively correspond to the voltages at the source and load ends of the line.

The corresponding open-ended transfer function (i.e., when $I_2(s) = 0$) is

$$H(s) = \frac{1}{\cosh\sqrt{sRC}}$$

Wilnai [28] considers the Laplace transform of the step response with magnitude V_0 for an open-ended distributed RC line,

$$V_2(s) = \frac{V_0}{s\cosh\sqrt{sRC}} \tag{2}$$

the inductive impedance (ωl), i.e., $r \ll \omega l$, so that the conventional LC-based analysis seems reasonable. However, with small feature sizes of thin-film and IC interconnects, we now find that $r \gg \omega l$ up to frequencies of $O(1)$ GHz; even at frequencies above $O(1)$ GHz, both terms are of comparable magnitude [13]. Thus, in the present regime of highly resistive interconnects, it seems natural to begin with an RC, rather than LC, model in obtaining the delay estimate.

While the RC-based perspective and the resulting diffusion equation have been noted by many authors, no closed form expression for the voltage response has previously been derived using appropriate boundary conditions. Such a pure time-domain solution is the central contribution of our work. Our analysis yields a simple analytical expression for the voltage response. Furthermore, though the solution of the diffusion equation does not refer directly to any wave propagation mechanism, we may yet consider reflections at discontinuities through the analogy of voltage propagation by electromagnetic vibrations (waves) to the propagation of heat waves [5], or via the Method of Images (or Reflections) [14].

To achieve a comparison with previous works, we study the open-ended RC line with ideal source and other idealized cases which have been treated in the literature. With the appropriate analysis of reflections, our pure time-domain solution for a finite-length open-ended line matches the solution derived from transform techniques [28, 23, 18]. This is in contrast to the previous time-domain approximate solution of Kaufman and Garrett [17], which does not behave properly for small values of t (and which is actually non-monotone in that region). We may also apply our delay analysis of the finite line to the case of arbitrary source and load impedances, again through consideration of reflections at the source and load discontinuities. Since our approach to computing the total response is based on summing various individual components of diffusion, each of which can be evaluated in the time domain, the total voltage can be easily estimated up to the required accuracy.

2 Previous Delay-Time Approximations for an RC Line

2.1 Lumped Models

Approximating the interconnect resistance and capacitance by lumped values R and C gives the time-domain response $v(t) = V_0(1 - e^{-\frac{t}{RC}})$ where V_0 is the input voltage applied. With this order partial differential equation of the form [9]:

$$\frac{\partial^2 v}{\partial x^2} = LC \cdot \frac{\partial^2 v}{\partial t^2}$$

and the solution to this wave equation is of the form

$$v(x) = A_1 e^{\theta x} + A_2 e^{-\theta x} \tag{1}$$

where $\theta \equiv$ propagation constant $\equiv \jmath\omega\sqrt{lc}$ (l and c are the inductance and capacitance per unit length, and ω is the wave frequency).

This model extends to *lossy* (RLC) interconnects by incorporating a series resistance. The same equations obtained for the lossless model can be used, with $Z_L = R + \jmath\omega L = \jmath\omega[\frac{R}{\jmath\omega} + L]$, i.e., $Z_L = \jmath\omega L'$ where $L' = \frac{R}{\jmath\omega} + L$ is the new inductance value. Similarly, one may incorporate a conductance G via $Z_C = G + \jmath\omega C = \jmath\omega[\frac{G}{\jmath\omega} + C]$, i.e., $Z_C = \jmath\omega C'$ where $C' = \frac{G}{\jmath\omega} + C$ is the new capacitance. The same solution derived for the lossless model can incorporate the new L' and C' values to capture attenuation in lossy lines.

Using the solution (1) to the wave equation, and the characteristic impedance of the line, one may treat the interconnect line as a 2-port and obtain equations for voltage and current at the terminal side of the 2-port in terms of voltage and current at the source side. This yields the 2-port matrix parameters, e.g., $ABCD$ parameters. To obtain the transient time-domain response of an interconnect line, the standard approach has been to calculate the response in the transform domain using 2-port parameters, and then apply inverse transforms to obtain the response in the time domain. We call this the *LC analysis*, or *wave equation*, approach. Since it may be complicated to apply the inverse transforms, various approximations are typically made which simplify the resulting expressions for the time-domain response.

In the well-studied analysis of an RC transmission line, the traditional LC analysis is extended to an RLC analysis after which L is set to zero. By contrast, if we initially model the interconnect as a pure distributed RC line, we obtain a *diffusion equation* (or *heat equation*) from which the solution for the transient response, depending on boundary conditions, can be calculated analytically. This *RC-based* delay analysis approach is the subject of the present paper.

Our motivation for adopting the RC-based delay analysis is as follows. For previous generation interconnects, such as for PCB, the resistance per unit length (r) is considerably smaller than

Delay Analysis of VLSI Interconnections Using the Diffusion Equation Model *

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Abstract

The traditional analysis of signal delay in a transmission line begins with a lossless LC representation, which yields a *wave equation* governing the system response; 2-port parameters are typically derived and the solution is obtained in the transform domain. In this paper, we begin with a distributed RC line model of the interconnect and analytically solve the resulting *diffusion equation* for the voltage response. A new closed form expression for voltage response is obtained by incorporating appropriate boundary conditions for interconnect delay analysis. We thus obtain the first exact, purely time-domain solution in the literature for the voltage response on the distributed RC line; in particular, we avoid the error that is inherent in the approximate approach of [17]. When appropriate analysis of reflections is used, the time-domain solution for a finite-length open-ended line matches the solution derived from transform techniques [28, 23, 18]. For various other configurations of distributed lines, our result can be easily extended to estimate the total voltage on the line. Finally, our discussion provides a unifying treatment of the past three decades of RC interconnect delay analyses. [This report has been revised to correct portions of the discussion in Section 4 and Appendix D.]

1 Overview

Delay analysis of VLSI interconnections is a key element in timing verification, gate-level simulation and performance-driven layout design. The standard approach to modeling interconnect delay has been based on a simple *lossless LC* model which considers only inductances (L) and capacitances (C). For this lossless model, the relationship between v and i gives rise to a second-

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